

# Computing a tree having a small vertex cover

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## Abstract

We consider a new Steiner tree problem, called vertex-cover-weighted Steiner tree problem. This problem defines the weight of a Steiner tree as the minimum weight of vertex covers in the tree, and seeks a minimum-weight Steiner tree in a given vertex-weighted undirected graph. Since it is included by the Steiner tree activation problem, the problem admits an  $O(\log n)$ -approximation algorithm in general graphs with  $n$  vertices. This approximation factor is tight up to a constant because it is NP-hard to achieve an  $o(\log n)$ -approximation for the vertex-cover-weighted Steiner tree problem on general graphs even if the given vertex weights are uniform and a spanning tree is required instead of a Steiner tree. In this paper, we present constant-factor approximation algorithms for the problem with unit disk graphs and with graphs excluding a fixed minor. For the latter graph class, our algorithm can be also applied for the Steiner tree activation problem.

## 1 Introduction

The problem of finding a minimum-weight tree in a graph has been extensively studied in the field of combinatorial optimization. A typical example is the Steiner tree problem in edge-weighted graphs; it has a long history of approximation algorithms, culminating in the currently best approximation factor of 1.39 [3, 11]. The Steiner tree problem has also been studied in vertex-weighted graphs, where the weight of a Steiner tree is defined as the total weight of the vertices spanned by the tree. We call this problem the *vertex-weighted Steiner tree problem* while the problem in the edge-weighted graphs is called the *edge-weighted Steiner tree problem*. There is an  $O(\log k)$ -approximation algorithm for the vertex-weighted Steiner tree problem with  $k$  terminals, and it is NP-hard to improve this factor because the problem includes the set cover problem [14].

In this paper, we present a new variation of the Steiner tree problem. Our problem is motivated by the following situation in communication networks. We assume that messages are exchanged along a tree in a network; this is the case in many popular routing protocols such as the spanning tree protocol [17]. We consider locating devices that will monitor the traffic in the tree. If a device is located at a vertex, it can monitor all the traffic that passes through links incident to that vertex. How many devices do we need for monitoring all of the traffic in the tree? Obviously, it depends on the topology of the tree. If the tree is a star, it suffices to locate one device at the center. If the tree is a path on  $n$  vertices, then it requires  $\lceil n/2 \rceil$  devices, because any vertex cover of the path consists of at least  $\lceil n/2 \rceil$  vertices. Our problem is to compute a tree that minimizes the number (or, more generally, the weight) of devices required to monitor all of the traffic.

More formally, our problem is defined as follows. Let  $G = (V, E)$  be an undirected graph associated with nonnegative vertex weights  $w \in \mathbb{R}_+^V$ . Throughout this paper, we will denote  $|V|$  by  $n$ . Let  $T \subseteq V$  be a set of vertices called *terminals*. The problem seeks a pair comprising a tree  $F$  and a vertex set  $U \subseteq V(F)$  such that (i)  $F$  is a Steiner tree with regard to the terminal set  $T$  (i.e.,  $T \subseteq V(F)$ ), and (ii)  $U$  is a vertex cover of  $F$  (i.e., each edge in  $F$  is incident to at least one vertex in  $U$ ). The objective is to find such a pair  $(F, U)$  that minimizes the weight  $w(U) := \sum_{v \in U} w(v)$  of the vertex cover. We call this the *vertex-cover-weighted (VC-weighted) Steiner tree problem*. We call the special case in which  $V = T$  the *vertex-cover-weighted (VC-weighted) spanning tree problem*. The aim of this paper is to investigate these fundamental problems.

Besides the motivation from the communication networks, there is another reason for the importance of the VC-weighted Steiner tree problem. The VC-weighted Steiner tree problem is a special case of the *Steiner tree activation problem*, which was formulated by Panigrahi [16]. In the Steiner tree activation problem, we are given a set  $W$  of nonnegative real numbers, and each edge  $uv$  in the graph is associated with an activation function  $f_{uv}: W \times W \rightarrow \{\top, \perp\}$ , where  $\top$  indicates that an edge  $uv$  is activated, and  $\perp$  indicates that it is not. A solution for the problem is defined as a  $|V|$ -dimensional vector  $x \in W^V$ . We say that a solution  $x$  *activates* an edge  $uv$  if  $f_{uv}(x(u), x(v)) = \top$ . The problem seeks a solution  $x$  that minimizes  $x(V) := \sum_{v \in V} x(v)$  subject to the constraint that the edges activated by  $x$  include a Steiner tree. In previous studies of this problem, an algorithm is allowed to run in polynomial time of  $|W|$ , and it is assumed that the activation function is monotone (i.e., if  $f_{uv}(i, j) = \top$ ,  $i \leq i'$ , and  $j \leq j'$ , then  $f_{uv}(i', j') = \top$ ). The Steiner tree activation problem models various natural settings in design of wireless networks [16]. To see that the Steiner tree activation problem includes the VC-weighted Steiner tree problem, define  $W$  as  $\{w(v) : v \in V\}$ , and let  $f_{uv}(i, j) = \top$  if and only if  $i \geq w(u)$  or  $j \geq w(v)$  for each edge  $uv$ . Under this setting, if  $x$  is a minimal vector that activates an edge set  $F$ , the objective  $x(V)$  is equal to the minimum weight of vertex covers of the subgraph induced by  $F$ . Hence the Steiner tree activation problem under this setting is equivalent to the VC-weighted Steiner tree problem.

The Steiner tree activation problem also contains the vertex-weighted Steiner tree problem. Indeed, vertex-weighted Steiner tree problem corresponds to the activation function  $f_{uv}$  such that  $f_{uv}(i, j) = \top$  if and only if  $i \geq w(u)$  and  $j \geq w(v)$  for each edge  $uv$ . Notice that the similarity of the activation functions for the VC-weighted and the vertex-weighted Steiner tree problems. Thus the VC-weighted Steiner tree problem is an interesting variant of the vertex-weighted Steiner tree and the Steiner tree activation problems, which are studied actively in the literature.

It is known that the Steiner tree activation problems admits an  $O(\log k)$ -approximation algorithm when  $|T| = k$ . Indeed, there is an approximation-preserving reduction from the problem to the vertex-weighted Steiner tree problem, and hence the  $O(\log k)$ -approximation algorithm for the latter problem implies that for the former problem. This approximation factor is proven to be tight even in the spanning tree variant of the problem [16].

Since the VC-weighted Steiner tree problem is included by the Steiner tree activation problem, the  $O(\log k)$ -approximation algorithm can also be applied to the VC-weighted problem. Moreover, Angel et al. [2] presented a reduction from the dominating set problem to the VC-weighted spanning tree problem with uniform vertex weights. This reduction implies that it is NP-hard to approximate the VC-weighted spanning tree problem within a factor of  $o(\log n)$  even if the given vertex weights are uniform. In Section 3, we present an alternative proof for this fact.

## 1.1 Our contributions

Because of the hardness of the VC-weighted spanning tree problem on general graphs, we will consider restricted graph classes. We show that the VC-weighted Steiner tree problem is NP-hard for unit disk graphs and planar graphs (Theorem 3). Moreover, we present constant-factor approximation algorithms for the problem with unit disk graphs (Corollary 2) and with graphs excluding a fixed minor (Theorem 8). Note that the later graph class contains planar graphs. For these graphs, it is known that the vertex-weighted

Steiner tree problem is NP-hard and admits constant-factor approximation algorithms [6, 21, 22]. Hence it is natural to investigate approximation algorithms for the VC-weighted Steiner tree problem in these graph classes. Moreover, unit disk graphs are regarded as a reasonable model of wireless networks, and the vertex-weighted Steiner tree problem in unit disk graphs has been actively studied in this context (see, e.g., [1, 13, 21, 22, 23]). Since our problem is motivated by an application in communication networks, it is reasonable to investigate the problem in unit disk graphs.

Our algorithm for unit disk graphs is based on a novel reduction to another optimization problem. The problem used in the reduction is similar to the connected facility location problem studied in [7, 19], but it is slightly different. In the connected facility location problem, we are given sets  $C, D \subseteq V$  of clients and facilities with an edge-weighted undirected graph  $G = (V, E)$ . If a facility  $f \in D$  is opened by paying an associated opening cost, any client  $i \in C$  can be allocated to  $f$  by paying the allocation cost, which is defined as the shortest path length from  $i$  to  $f$  multiplied by the demand of  $i$ . The opened facilities must be spanned by a Steiner tree, which incurs a connection cost defined as the edge weight of the tree. The objective is to find a set of opened facilities and a Steiner tree connecting them, that minimizes the sum of the opening cost, the allocation cost, and the connection cost. Our problem differs from the connected facility location problem in the fact that each client  $i$  can be allocated to an opened facility  $f$  only when  $i$  is adjacent to  $f$  in  $G$ , and there is no cost for the allocation. It can be regarded as a combination of the dominating set and the edge-weighted Steiner tree problems. Hence we call this the *connected dominating set problem*, although in the literature, this name is usually reserved for the case where the connection cost is defined by vertex weights and all vertices in the graph are clients. From a geometric property of unit disk graphs, we show that our reduction preserves the approximation guarantee up to a constant factor if the graph is a unit disk graph (Theorem 1). To solve the connected dominating set problem, we present a linear programming (LP) rounding algorithm. This algorithm relies on an idea presented by Huang, Li, and Shi [13], who considered a variant of the connected dominating set problem in unit disk graphs. Although their algorithm is only for minimizing the number of vertices in a solution, we prove that it can be extended to our problem.

For graphs excluding a fixed minor, we solve the VC-weighted Steiner tree problem by presenting a constant-factor approximation algorithm for the Steiner tree activation problem. Our algorithm simply combines the reduction to the vertex-weighted Steiner tree problem and the algorithm of Demaine, Hajiaghayi, and Klein [6] for the vertex-weighted Steiner tree problem in graphs excluding a fixed minor. However, analyzing it is not straightforward, because the reduction does not preserve the minor-freeness of the input graphs. Nevertheless, we show that the algorithm of Demaine et al. achieves a constant-factor approximation for the graphs constructed by the reduction (Section 5).

## 1.2 Organization

The remainder of this paper is organized as follows. Section 2 introduces the notation and preliminary facts used throughout the paper. Section 3 presents hardness results on the VC-weighted Steiner tree problem. Sections 4 and 5 provide constant-factor approximation algorithms for unit disk graphs and for graphs excluding a fixed minor, respectively. Section 6 concludes the paper.

## 2 Preliminaries

We first define the notation used in this paper. Let  $G = (V, E)$  be a graph with the vertex set  $V$  and the edge set  $E$ . We sometimes identify the graph  $G$  with its edge set  $E$  and by  $V(G)$  denote the vertex set of  $G$ . When  $G$  is a tree,  $L(G)$  denotes the set of leaves of  $G$ .

Let  $U$  be a subset of  $V$ . Then  $G - U$  denotes the subgraph of  $G$  obtained by removing all vertices in  $U$  and all edges incident to them.  $G[U]$  denotes the subgraph of  $G$  induced by  $U$ .

We denote a singleton vertex set  $\{v\}$  by  $v$ . An edge joining vertices  $u$  and  $v$  is denoted by  $uv$ . For a vertex  $v$ ,  $N_G(v)$  denotes the set of neighbors of  $v$  in a graph  $G$ , i.e.,  $N_G(v) = \{u \in V : uv \in E\}$ .  $N_G[v]$

indicates  $N_G(v) \cup v$ . We let  $d_G(v)$  denote  $|N_G(v)|$ . For a set  $U$  of vertices,  $N_G(U)$  denotes  $(\bigcup_{v \in U} N_G(v)) \setminus U$ . When the graph  $G$  is clear from the context, we may remove the subscripts from our notation. We say that a vertex set  $U$  *dominates* a vertex  $v$  if  $v \in U$ , or  $U$  contains a vertex  $u$  that is adjacent to  $v$ . If a vertex set  $U$  dominates each vertex  $v$  in another vertex set  $W$ , then we say that  $U$  dominates  $W$ .

A graph  $G$  is a unit disk graph when there is an embedding of the vertex set into the Euclidean plane such that two vertices  $u$  and  $v$  are joined by an edge if and only if their Euclidean distance is at most 1. If  $G$  is a unit disk graph, we call such an embedding a *geometric representation* of  $G$ .

Let  $G$  and  $H$  be undirected graphs. We say that  $H$  is a *minor* of  $G$  if  $H$  is obtained from  $G$  by deleting edges and vertices and by contracting edges. If  $H$  is not a minor of  $G$ ,  $G$  is called  *$H$ -minor-free*. By Kuratowski's theorem, a graph is planar if and only if it is  $K_5$ -minor-free and  $K_{3,3}$ -minor-free.

As mentioned in Section 1, the Steiner tree activation problem contains both the VC-weighted and the vertex-weighted Steiner tree problems. In addition, the Steiner tree activation problem can be reduced to the vertex-weighted Steiner tree problem, as summarized in the following theorem.

**Theorem 1.** *There is an approximation-preserving reduction from the Steiner tree activation problem to the vertex-weighted Steiner tree problem. Hence, if the latter problem admits an  $\alpha$ -approximation algorithm, the former problem does also.*

*Proof.* Recall that an instance  $I$  of the Steiner tree activation problem consists of an undirected graph  $G = (V, E)$ , a terminal set  $T$ , a range  $W \subseteq \mathbb{R}_+$ , and an activation function  $f_{uv}: W \times W \rightarrow \{\top, \perp\}$  for each  $uv \in E$ . We define a copy  $v_i$  of a vertex  $v$  for each  $v \in V$  and  $i \in W$ , and associate  $v_i$  with the weight  $w(v_i) := i$ . We join  $u_i$  and  $v_j$  by an edge if  $uv \in E$  and  $f_{uv}(i, j) = \top$ . In addition, we join each terminal  $t \in T$  with its copies  $t_i$ ,  $i \in W$ . The weight  $w(t)$  of  $t$  is defined to be 0. Let  $G'$  be the obtained graph on the vertex set  $T \cup \{v_i: v \in V, i \in W\}$ . Let  $I'$  be the instance of the vertex weighted Steiner tree problem that consists of the graph  $G'$ , the vertex weights  $w$ , and the terminal set  $T$ . From an inclusion-wise minimal Steiner tree  $F$  feasible to  $I'$ , define a vector  $x \in W^V$  by  $x(v) = \max\{i \in W: v_i \in V(F)\}$  for each  $v \in V$ . Then  $x$  activates a Steiner tree in the original instance  $I$ , and  $x(V)$  is equal to the vertex weight of  $F$ . Hence there is a one-to-one correspondence between a minimal Steiner tree in  $I'$  and a feasible solution in  $I$ , and they have the same objective values in their own problems. Hence the above reduction is an approximation-preserving reduction from the Steiner tree activation problem to the vertex-weighted Steiner tree problem.  $\square$

We do not claim the originality of Theorem 1; we believe that this reduction has been already known although we are aware of no previous study describing this reduction explicitly.

We note that the reduction claimed in Theorem 1 transforms the input graph, and hence it may not be closed in a graph class. In fact, we can observe that the reduction is not closed in unit disk graphs or planar graphs.

### 3 Hardness of VC-weighted spanning tree and Steiner tree problems

In this section, we present hardness results of the VC-weighted spanning tree and Steiner tree problems. First, we prove that it is NP-hard to approximate the VC-weighted spanning tree problem within a factor of  $o(\log n)$  even if vertex weights are uniform. This fact has already been proven by Angel et al. [2]. Here, we give an alternative proof which consists of an approximation-preserving reduction from the set cover problem.

**Theorem 2.** *It is NP-hard to approximate VC-weighted spanning tree problem within a factor of  $o(\log n)$  even if the given vertex weights are uniform.*

*Proof.* Recall that an instance of the set cover problem consists of a finite set  $S$  and a family  $\mathcal{X}$  of subsets of  $S$ . The objective of the problem is to find a subfamily  $\mathcal{X}'$  of  $\mathcal{X}$  such that  $\bigcup_{X \in \mathcal{X}'} X = S$  and  $|\mathcal{X}'|$  is minimized. A feasible solution for the set cover instance is called a set cover.

For each set  $X \in \mathcal{X}$ , we define a vertex  $v_X$  corresponding to it. Let  $W$  denote  $\{v_X : X \in \mathcal{X}\}$ . The set cover instance defines a bipartite graph on the vertex set  $S \cup W$ ; two vertices  $v \in S$  and  $v_X \in W$  are joined by an edge if and only if  $v \in X$ . To this bipartite graph, we add edges so that any two vertices in  $W$  are adjacent. Let  $G = (S \cup W, E)$  be the obtained graph. We reduce the set cover instance to the instance of VC-weighted spanning tree problem on this graph with uniform vertex weights. We prove that this reduction is approximation-preserving. Since it is NP-hard to approximate the set cover instance within a factor of  $o(\log |S|)$  [18], this completes the proof.

Let  $\mathcal{X}'$  be a set cover for instance  $(S, \mathcal{X})$ . We define  $W' := \{v_X \in W : X \in \mathcal{X}'\}$  from  $\mathcal{X}'$ . Since  $\mathcal{X}'$  is a set cover, for each  $u \in S$ , there exists  $v_X \in W'$  that is adjacent to  $u$  in  $G$ . We define  $F$  as the set of edges joining such pairs of  $u \in S$  and  $v_X \in W'$ . Let  $F'$  be a star on  $W$  such that its center is an arbitrary vertex in  $W$  and  $F'$  spans all vertices in  $W$ . Then  $F \cup F'$  is a spanning tree in  $G$ , and  $W'$  is a vertex cover in  $F \cup F'$ . Thus, for any set cover  $\mathcal{X}'$ , there exists a feasible solution  $(F \cup F', W')$  with  $|W'| = |\mathcal{X}'|$  for the instance of VC-weighted spanning tree problem.

Let us consider the other direction. Let  $(F, U)$  be a solution for the instance of VC-weighted spanning tree problem. If  $U \subseteq W$ , then  $\mathcal{X}' := \{X \in \mathcal{X} : v_X \in U\}$  is a set cover in  $(S, \mathcal{X})$  because each vertex  $v \in S$  is adjacent to a vertex  $v_X \in U$  in  $F$ , and hence  $\mathcal{X}'$  contains a set  $X$  with  $v \in X$ . Notice that  $|U| = |\mathcal{X}'|$ .

Suppose that  $U$  contains a vertex  $u \in S$ . Let  $v_{X_1}, \dots, v_{X_k}$  be the vertices in  $W$  that are adjacent to  $u$  on  $F$ . We prove that the solution  $(F, U)$  can be modified to another feasible solution  $(F', U')$  with  $|U' \cap S| < |U \cap S|$  and  $|U'| \leq |U|$ . By repeating this modification, we obtain a feasible solution whose vertex cover is contained by  $W$ . We define  $U'$  as  $(U \setminus u) \cup v_{X_1}$ . Let  $F'$  be the edge set obtained from  $F$  by replacing all edges  $uv_{X_2}, \dots, uv_{X_k}$  with  $v_{X_1}v_{X_2}, \dots, v_{X_1}v_{X_k}$ . Then  $F'$  is a spanning tree, and  $U'$  is a vertex cover on  $F'$ .  $\square$

As noted in Theorem 1, there is an approximation-preserving reduction from the Steiner tree activation problem to the vertex-weighted Steiner tree problem, and the latter problem admits an  $O(\log |T|)$ -approximation algorithm in general graphs. Since the Steiner tree activation problem includes VC-weighted Steiner tree problem, this indicates that VC-weighted Steiner tree problem also admits an  $O(\log |T|)$ -approximation algorithm. By Theorem 2, the approximation factor achieved by this algorithm is tight up to a constant.

Next, we consider unit disk graphs and planar graphs. We show that the VC-weighted Steiner tree problem is NP-hard for these graph classes.

**Theorem 3.** *VC-weighted Steiner tree problem is NP-hard for unit disk graphs and for planar graphs.*

*Proof.* Garey and Johnson [9] proved that the edge-weighted Steiner tree problem is NP-hard even on the grid graphs. We show that the edge-weighted Steiner tree problem on grid graphs can be reduced to the VC-weighted Steiner tree problem on unit disk graphs. We can suppose without loss of generality that the distance between every two adjacent vertices  $u$  and  $v$  in the grid graph is 4. For each adjacent vertices  $u$  and  $v$ , we subdivide the edge  $uv$  by adding three new vertices  $i, j$ , and  $k$  distributed equally between  $u$  and  $v$  as illustrated in Figure 1. The graph obtained by this way is a unit disk graph because two vertices in the graph are adjacent if and only if the distance between them is exactly a unit length. From the edge weights  $w'$  of the original graph, we define the vertex weights  $w$  of the new graph by  $w(u) = w(v) = 0$ ,  $w(i) = w(k) = +\infty$ , and  $w(j) = w'(ij)$ . Then, if edges  $ui, ij, jk$ , and  $kv$  are included in a Steiner tree, a minimum-weight vertex cover on the tree includes  $u, v$ , and  $j$ . Hence, the minimum weight of vertex covers on a Steiner tree in the unit disk graph is equal to the edge weight of the corresponding Steiner tree in the original graph. Hence this gives an approximation-preserving reduction from the edge-weighted Steiner tree problem on grid graphs to the VC-weighted Steiner tree problem on unit disk graphs.

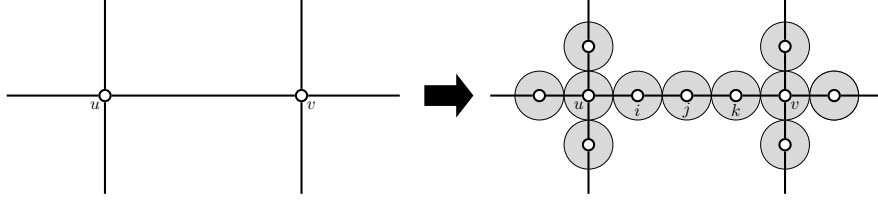


Figure 1: Reduction from the edge-weighted Steiner tree problem on grid graphs to the VC-weighted Steiner tree problem on planar unit disk graphs

Notice that the graph constructed by the above reduction is also planar. Hence this also proves the NP-hardness of VC-weighted Steiner tree problem on planar graphs.  $\square$

## 4 VC-weighted Steiner tree problem in unit disk graphs

The aim of this section is to present a constant-factor approximation algorithm for the VC-weighted Steiner tree problem in unit disk graphs. Our algorithm consists of two steps. In the first step, we reduce the VC-weighted Steiner tree problem to another optimization problem, which is called the connected dominating set problem. We present this reduction in Section 4.1. When the original problem is the VC-weighted spanning tree problem, the connected dominating set problem can be solved by a simpler algorithm, which we will explain in Section 4.2. Then, in Section 4.3, we will present an LP-rounding algorithm for the general case of the connected dominating set problem.

### 4.1 Reduction

As noted in Theorem 1, the Steiner tree activation problem can be reduced to the vertex-weighted Steiner tree problem. Since the VC-weighted Steiner tree problem is included in the Steiner tree activation problem, the reduction also applies to the VC-weighted Steiner tree problem. Since there is a constant-factor approximation algorithm for the vertex-weighted Steiner tree problem in unit disk graphs, this reduction gives a constant-factor approximation for the VC-weighted problem if the graph constructed by the reduction is a unit disk graph. However, the constructed graph may not be a unit disk graph, even if the original graph is a unit disk graph. This can be seen through an example. Let  $G = (V, E)$  be a star graph with the center vertex  $v$ . We do not specify the terminal set  $T$ , because it is not important here. When the original problem is the VC-weighted Steiner tree problem, the reduction given in Theorem 1 can be simplified as follows. Two copies  $u_{\circ}$  and  $u_{\bullet}$  are constructed from each vertex  $u \in V(G)$ , where  $u_{\circ}$  indicates that  $u$  is not included in the vertex cover of the solution, and  $u_{\bullet}$  indicates that it is. For each edge  $uu' \in E$ , the graph constructed by the reduction contains edges  $u_{\bullet}u'_{\bullet}$ ,  $u_{\circ}u'_{\bullet}$ , and  $u_{\bullet}u'_{\circ}$ . Moreover, each terminal  $t \in T$  is adjacent to its copies  $t_{\circ}$  and  $t_{\bullet}$ . Let  $G'$  be a graph on the vertex set  $T \cup \{u_{\circ}, u_{\bullet} : u \in V\}$  that is constructed in this way. By solving the vertex-weighted Steiner tree problem on  $G'$ , we can compute a solution to the VC-weighted problem on  $G$ . If the degree of  $v$  is at most 5,  $G$  is a unit disk graph. The degree of  $v_{\bullet}$  in  $G'$  is twice the degree of  $v$  in  $G$ , and any two neighbors of  $v_{\bullet}$  are not adjacent in  $G'$ . Hence  $G'$  contains  $K_{1,6}$  as an induced subgraph if the degree of  $v$  in  $G$  is at least 3. Since no unit disk graph contains  $K_{1,6}$  as an induced subgraph, this means that  $G'$  is not a unit disk graph.

Our idea is to reduce the VC-weighted Steiner tree problem to another optimization problem. This is inspired by a constant-factor approximation algorithm for the vertex-weighted Steiner tree problem on a unit disk graph [22, 21]. This algorithm is based on a reduction from the vertex-weighted to the edge-weighted Steiner tree problems. The reduction is possible because the former problem always admits an optimal

Steiner tree in which the maximum degree is a constant if the graph is a unit disk graph. Even in the VC-weighted Steiner tree problem, if there is an optimal solution  $(F, U)$  such that the maximum degree of vertices in the vertex cover  $U$  is a constant in the Steiner tree  $F$ , then we can reduce the problem to the edge-weighted Steiner tree problem. However, there is an instance of the VC-weighted Steiner tree problem that admits no such optimal solution. For example, if the vertex weights are uniform, and the graph includes a star in which all of the terminals are its leaves, then the star is the Steiner tree in the optimal solution, and its minimum vertex cover consists of only the center of the star. The degree of the center of the star is not bounded by a constant. Hence it seems that it would be difficult to reduce the VC-weighted Steiner tree problem to the edge-weighted problem.

We reduce the VC-weighted Steiner tree problem to a problem similar to the connected facility location problem. The reduction is based on a geometric property of unit disk graphs, and we will begin by proving this property. The following lemma gives a basic claim about geometry. For two points  $i$  and  $j$  on the plane, we denote their Euclidean distance by  $l_{ij}$ .

**Lemma 1.** *Let  $i$  be a point on the Euclidean plane, and let  $\alpha \in (1/2, 3/4]$ . Let  $P$  be a set of points on the plane such that  $\alpha < l_{ik}/l_{ij} \leq 1/\alpha$  holds for all  $j, k \in P$ . If  $|P| > 2\pi/\arccos(\alpha/2 + 3/(8\alpha))$ , then there exist  $j, k \in P$  such that  $l_{jk} < \max\{l_{ij}, l_{ik}\}/2$ .*

*Proof.* Since  $|P| > 2\pi/\arccos(\alpha/2 + 3/(8\alpha))$ , there exist  $j, k \in P$  such that  $\theta := \angle jik < \arccos(\alpha/2 + 3/(8\alpha))$ . We note that  $l_{jk}^2 = l_{ij}^2 + l_{ik}^2 - 2l_{ij}l_{ik}\cos\theta$ . Without loss of generality, we assume  $l_{ij} \geq l_{ik}$ . Then,  $(\max\{l_{ij}, l_{ik}\})^2 = l_{ij}^2$ . Hence it suffices to show that  $-4l_{ik}^2 - 3l_{ij}^2 + 8l_{ij}l_{ik}\cos\theta > 0$ .

Let  $\beta := l_{ik}/l_{ij}$ . Then,  $\alpha < \beta \leq 1$  holds.  $\max_{\alpha < \beta \leq 1} 4\beta + 3/\beta = 4\alpha + 3/\alpha$  holds, where the maximum is attained by  $\beta = \alpha$ . Hence the required inequality is verified by

$$\begin{aligned} -4l_{ik}^2 - 3l_{ij}^2 + 8l_{ij}l_{ik}\cos\theta &= l_{ij}l_{ik} \left( -4\beta - \frac{3}{\beta} + 8\cos\theta \right) \\ &\geq l_{ij}l_{ik} \left( -4\alpha - \frac{3}{\alpha} + 8\cos\theta \right) \\ &= 0. \end{aligned}$$

□

Our reduction requires the assumption that there is an optimal solution  $(F, U)$  for the VC-weighted Steiner tree problem such that the degree of each vertex  $v \in U$  is bounded by a constant  $\alpha$  in the tree  $F - (L(F) \setminus U)$ . The following lemma proves that the assumption holds with  $\alpha = 29$  if the input graph is a unit disk graph.

**Lemma 2.** *If the input graph  $G = (V, E)$  is a unit disk graph, the VC-weighted Steiner tree problem admits an optimal solution consisting of a Steiner tree  $F$  and a vertex cover  $U$  of  $F$  such that the degree of each vertex in  $U$  is at most 29 in  $F - (L(F) \setminus U)$ .*

*Proof.* For two vertices  $u, v \in V$ , let  $l_{uv}$  denote the Euclidean distance between  $u$  and  $v$  in the geometric representation of  $G$ . Let  $(F, U)$  be an optimal solution for the VC-weighted Steiner tree problem. Without loss of generality, we can assume that  $(F, U)$  satisfies the following conditions:

- (a)  $(F, U)$  maximizes  $|L(F) \setminus U|$  over all optimal solutions;
- (b)  $F$  minimizes  $\sum_{e \in F} l_e$  over all optimal solutions subject to (a);
- (c)  $(F, U)$  minimizes the number of vertices  $v \in U$  such that  $|\{u \in U : uv \in F\}| \geq 6$  over all optimal solutions subject to (a) and (b).

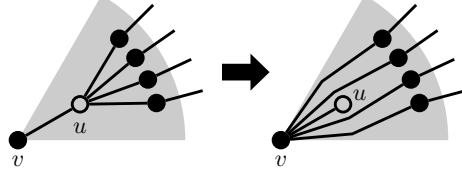


Figure 2: Transformation of  $F$  when  $l_{vu'} \leq 1$  for all  $u' \in A_u$

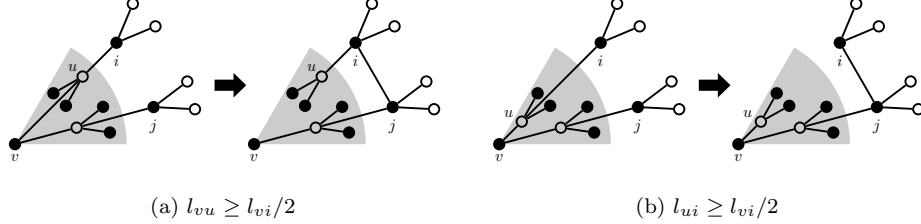


Figure 3: Transformation of a tree  $F$  when  $l_{ij} < \max\{l_{vi}, l_{vj}\}/2$

Let  $v \in U$ . Let  $M_v := \{u \in U : uv \in F\}$  and  $M'_v := \{u \in V \setminus (U \cup L(F)) : uv \in F\}$ . We prove the lemma by showing that  $|M_v| \leq 5$  and  $|M'_v| \leq 24$ .

We first show that  $|M_v| \leq 5$ . Suppose that there are two distinct vertices  $i, j \in M_v$  such that  $l_{ij} < \max\{l_{vi}, l_{vj}\}$ . Without loss of generality, let  $l_{vi} = \max\{l_{vi}, l_{vj}\}$ , and denote  $F \setminus \{vi\} \cup \{ij\}$  by  $F'$ . Then  $F'$  is a Steiner tree,  $U$  is a vertex cover of  $F'$ ,  $L(F) \setminus U = L(F') \setminus U$ , and  $\sum_{e \in F'} l_e < \sum_{e \in F} l_e$ . Since the existence of such an  $F'$  contradicts condition (b),  $M_v$  contains no such vertices  $i$  and  $j$ .

If  $|M_v| \geq 7$ , there must be two vertices  $i, j \in M_v$  such that  $\angle ivj < \pi/3$ , and  $l_{ij} < \max\{l_{vi}, l_{vj}\}$  holds for these vertices. Hence  $|M_v| \leq 6$  holds. Suppose that  $|M_v| = 6$ . In this case,  $M_v = \{u_1, \dots, u_6\}$ , and  $l_{vu_k} = l_{vu_{k+1}} = l_{u_k u_{k+1}}$  holds for all  $k = 1, \dots, 6$ , where, for notational convenience, we let  $u_7$  denote  $u_1$ . If  $|M_{u_1}| \leq 4$ , we define  $F'$  as  $F \setminus \{vu_2\} \cup \{u_1 u_2\}$ . Then,  $F'$  is a Steiner tree,  $U$  is a vertex cover of  $F'$ ,  $L(F') \setminus U = L(F) \setminus U$ , and  $\sum_{e \in F'} l_e = \sum_{e \in F} l_e$ . Replacing  $F$  by  $F'$  decreases the number of vertices  $v \in U$  such that  $|M_v| \geq 6$ , which contradicts condition (c). If  $|M_{u_1}| \geq 5$ , then (i) there exist  $i, j \in M_{u_1} \setminus v$  such that  $l_{ij} < \max\{l_{u_1 i}, l_{u_1 j}\}$ , or (ii) there exist  $i \in M_{u_1} \setminus v$  and  $j \in \{v, u_2, u_6\}$  such that  $l_{ij} < \max\{l_{u_1 i}, l_{u_1 j}\}$ . Case (i) contradicts condition (b), as observed above. In case (ii), we define  $F'$  as  $F \setminus \{u_1 v\} \cup \{ij\}$  if  $\max\{l_{u_1 i}, l_{u_1 j}\} = l_{u_1 i}$ , and as  $F \setminus \{u_1 v\} \cup \{ij\}$  if  $\max\{l_{u_1 i}, l_{u_1 j}\} = l_{u_1 j}$ . In either case,  $F'$  is a Steiner tree,  $U$  is a vertex cover of  $F'$ ,  $L(F') \setminus U = L(F) \setminus U$ , and  $\sum_{e \in F'} l_e > \sum_{e \in F} l_e$ , which contradicts condition (b). Hence  $|M_v| \leq 5$  holds.

Next, we prove  $|M'_v| \leq 24$ . Let  $u \in M'_v$ . Since  $u$  is not a leaf,  $u$  has a neighbor other than  $v$ . We denote by  $A_u$  the set of neighbors of  $u$  other than  $v$ . Since  $u \notin U$ , each vertex in  $A_u$  is included in  $U$ . If  $l_{vu'} \leq 1$  holds for all vertices  $u' \in A_u$ , consider  $F' := F \setminus (vu \cup \{uu' : u' \in A_u\}) \cup \{vu' : u' \in A_u\}$ . Then,  $F'$  is a Steiner tree,  $U$  is a vertex cover of  $F'$ , and  $L(F') \setminus U = (L(F) \setminus U) \cup \{u\}$  (see Figure 2). Since the existence of such an  $F'$  contradicts condition (a), there is at least one vertex  $u' \in A_u$  with  $l_{vu'} > 1$ . We choose one of these vertices for each  $u \in M'_v$ , and let  $B$  denote the set of those chosen vertices (hence  $B$  includes exactly one vertex in  $A_u$  for each  $u \in M'_v$ ).

Suppose there exist two vertices  $i, j \in B$  such that  $l_{ij} < \max\{l_{vi}, l_{vj}\}/2$ . Let  $l_{vi} = \max\{l_{vi}, l_{vj}\}$ . Let  $u$  denote the common neighbor of  $v$  and  $i$ . Then,  $l_{vu}$  or  $l_{ui}$  is at least  $l_{vi}/2$ . If  $l_{vu} \geq l_{vi}/2$ , then replace edge  $vu$  by  $ij$  in  $F$  (see Figure 3(a)). Otherwise, replace edge  $ui$  by  $ij$  in  $F$  (see Figure 3(b)). Let  $F'$  denote the tree obtained by this replacement.  $F'$  is a Steiner tree,  $U$  is a vertex cover of  $F'$ ,  $L(F) \setminus U \subseteq L(F') \setminus U$ , and  $\sum_{e \in F'} l_e < \sum_{e \in F} l_e$  hold. Since this contradicts condition (a) or (b), there exists no such pair of vertices  $i, j \in B$ .



We divide  $B$  into  $B' := \{i \in B \mid l_{vi} \leq 1.41\}$  and  $B'' := \{i \in B \mid 1.41 \leq l_{vi}\}$ . Notice that  $1/1.41 \leq l_{vi}/l_{vj} \leq 1.41$  holds for any  $i, j \in B'$ . Hence, by Lemma 1,  $|B'| \leq \lfloor 2\pi / \arccos(1/2.82 + 4.23/8) \rfloor = 12$ . Moreover,  $3/5 \leq l_{vi}/l_{vj} \leq 5/3$  holds for any  $i, j \in B''$ . Hence, by Lemma 1,  $|B''| \leq \lfloor 2\pi / \arccos(3/10 + 5/8) \rfloor = 12$ . Since  $|M'_v| \leq |B| = |B'| + |B''| \leq 24$ , this proves the lemma.  $\square$

In the remainder of this subsection, we assume that  $G$  is not necessarily a unit disk graph, but there is an optimal solution  $(F, U)$  for the VC-weighted Steiner tree problem such that the degree of each vertex  $v \in U$  is at most a constant  $\alpha$  in the tree  $F - (L(F) \setminus U)$ . Based on this assumption, we reduce the VC-weighted Steiner tree problem to another optimization problem. First, let us define the problem used in the reduction.

**Definition 1** (Connected dominating set problem). *Let  $G = (V, E)$  be an undirected graph, and let  $T \subseteq V$  be a set of terminals. Each edge  $e$  is associated with the length  $l(e) \in \mathbb{R}_+$ , each vertex  $v$  is associated with the weight  $w(v) \in \mathbb{R}_+$ , and  $l(e) \leq \min\{w(u), w(v)\}$  holds for each edge  $e = uv \in E$ . The problem seeks a pair of a tree  $F \subseteq E$  and a vertex set  $S \subseteq V$  such that  $S$  dominates  $T$  and  $F$  spans  $S$ . Let  $l(F)$  denote  $\sum_{e \in F} l(e)$ . The objective is to minimize  $w(S) + l(F)$ .*

We note that there are several previous studies of the connected dominating set problem [12, 5, 1, 23]. However, the algorithms in those studies do not apply to our setting because they consider only the case  $T = V$ .

**Theorem 4.** *If there is a  $\beta$ -approximation algorithm for the connected dominating set problem in a graph  $G$ , then there is an  $(\alpha + 1)\beta$ -approximation algorithm for the VC-weighted Steiner tree problem with input graph  $G$ .*

*Proof.* Suppose that an instance  $I$  of the VC-weighted Steiner tree problem consists of an undirected graph  $G = (V, E)$ , a terminal set  $T \subseteq V$ , and vertex weights  $w \in \mathbb{R}_+^V$ . We define the edge length  $l(e)$  as  $\min\{w(u), w(v)\}$  for each  $e = uv \in E$ , and define an instance  $I'$  of the connected dominating set problem from  $G$ ,  $T$ ,  $w$ , and  $l$ . We show that the optimal objective value of  $I'$  is at most  $\alpha + 1$  times that of  $I$ , and a feasible solution for  $I$  can be constructed from the one for  $I'$  without increasing the objective value. Combined with the  $\beta$ -approximation algorithm for  $I'$ , these claims give an  $(\alpha + 1)\beta$ -approximation algorithm for  $I$ .

First, we prove that the optimal objective value of  $I'$  is at most  $\alpha + 1$  times that of  $I$ . Let  $(F, U)$  be an optimal solution for  $I$ . Then, the optimal objective value of  $I$  is  $w(U)$ . Since  $F$  spans  $T$  and  $U$  is a vertex cover of  $F$ ,  $U$  dominates  $T$ . Define  $F' := F - (L(F) \setminus U)$ . Since  $F'$  is a tree spanning  $U$ ,  $(F', U)$  is a feasible solution for  $I'$ . If  $e = uv \in F'$ , then  $u$  or  $v$  is included in  $U$ , and  $l(e)$  is at most  $w(u)$  and  $w(v)$ . Hence  $l(F') \leq \sum_{v \in U} w(v) d_{F'}(v)$ . By assumption,  $d_{F'}(v) \leq \alpha$  holds for each  $v \in U$ . Hence  $l(F') \leq \alpha w(U)$ . Since the objective value of  $(F', U)$  in  $I'$  is  $l(F') + w(U)$ , the optimal objective value of  $I'$  is at most  $(\alpha + 1)w(U)$ .

Next, we prove that a feasible solution  $(F, S)$  for  $I'$  provides a feasible solution for  $I$ , and its objective value is at most that of  $(F, S)$ . Since  $S$  dominates  $T$ , if a terminal  $t \in T$  is not spanned by  $F$ , there is a vertex  $v \in S$  with  $tv \in E$ . We let  $F'$  be the set of such edges  $tv$ . Notice that  $F \cup F'$  is a Steiner tree of the terminal set  $T$ . For each edge  $e \in F$ , choose an end vertex  $v$  of  $e$  such that  $l(e) = w(v)$ . Let  $S'$  denote this set of chosen vertices. Then,  $S' \cup S$  is a vertex cover of  $F \cup F'$ . Hence  $(F \cup F', S' \cup S)$  is feasible for  $I$ . Since  $w(S' \cup S) \leq w(S) + l(F)$ , the objective value of  $(F \cup F', S' \cup S)$  is at most that of  $(F, S)$ .  $\square$

## 4.2 Algorithm for the connected dominating set problem with $T = V$

In the remainder of this section, we present algorithms for the connected dominating set problem. As a warm-up, we will first discuss the case  $T = V$ , which arises in the reduction from the VC-weighted spanning tree problem. We show that the problem admits a simple constant-factor approximation algorithm for any graphs in which the minimum-weight dominating set problem admits a constant-factor approximation. This class includes unit disk graphs [23, 8]. Below, we let  $\beta$  denote the approximation factor for the minimum-weight dominating set problem.

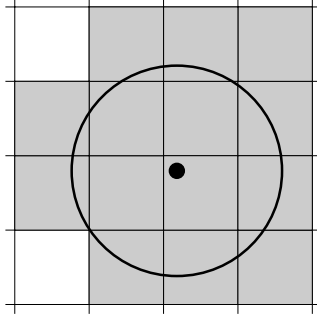


Figure 4: A unit disk and squares of sides  $\sqrt{2}/2$

Our algorithm first computes a  $\beta$ -approximate solution  $S$  of the minimum dominating set of the graph. Then, it computes a  $\beta'$ -approximation of the minimum edge-weighted Steiner tree that spans  $S$ . Let  $F$  denote the computed Steiner tree. Then,  $(F, S)$  is our approximate solution for the connected dominating set problem.

**Theorem 5.** *The above algorithm achieves a  $\beta(\beta' + 1)$ -approximation for the connected dominating set problem.*

*Proof.* Let  $(F, S)$  denote a solution output by the algorithm, and let  $(F^*, S^*)$  be an optimal solution for the problem. Since  $S^*$  is a dominating set of  $G$ ,  $w(S) \leq \beta w(S^*)$  holds by the definition of  $S$ . Also,  $F^*$  spans  $S^*$ . Hence, if a vertex  $v \in S$  is included in  $S^*$ ,  $v$  is spanned by  $F^*$ . If  $v$  is not included in  $S^*$ , there is a vertex  $v' \in S^*$  that is adjacent to  $v$ . Let  $F'$  denote a set of such edges  $vv'$ . Then  $F^* \cup F'$  is a connected subgraph that spans  $S$ . Notice that all edges in  $F'$  are incident to some vertices in  $S$ , and the degree of each vertex in  $S$  is at most one in  $F'$ . Hence  $l(F') \leq w(S)$ . Therefore,  $l(F) \leq \beta' l(F^* \cup F') \leq \beta' (l(F^*) + w(S))$ . The objective value of the solution  $(F, S)$  is  $l(F) + w(S) \leq \beta' l(F^*) + (\beta' + 1)w(S) \leq \beta' l(F^*) + (\beta' + 1)\beta w(S^*)$ . Hence its approximation factor is at most  $\beta(\beta' + 1)$ .  $\square$

As mentioned above, in a unit disk graph,  $\beta$  is a constant. The edge-weighted Steiner tree problem admits a constant-factor approximation algorithm for any graphs [11, 3]. Hence Theorem 5 provides the following corollary.

**Corollary 1.** *The VC-weighted spanning tree problem admits a constant-factor approximation algorithm in unit disk graphs.*

### 4.3 Algorithm for the connected dominating set problem

We now provide a constant-factor approximation algorithm for the general case of the connected dominating set problem in unit disk graphs. Our algorithm is based on the idea given by Huang, Li, and Shi [13].

We say that a graph  $G = (V, E)$  has a property  $\pi$  if there is a partition  $\Pi := \{V_1, \dots, V_k\}$  of the vertex set  $V$  such that each  $V_i \in \Pi$  induces a clique in  $G$  and each vertex  $v \in V$  satisfies  $|\{i = 1, \dots, k : V_i \cap N_G[v] \neq \emptyset\}| \leq \theta$  with some constant  $\theta$ . The following lemma proves that a unit disk graph possesses this property.

**Lemma 3.** *A unit disk graph has the property  $\pi$  with a constant  $\theta = 14$ .*

*Proof.* Divide the Euclidean plane into squares of side  $\sqrt{2}/2$ . We define each class  $V_i$  of the partition  $\mathcal{P}$  as a set of vertices whose positions are on the same square in the geometric representation of  $G$ . If a vertex  $u$  is on a side with more than one square, then we assign  $u$  to the upper-right square. Then, since any two vertices in the same square are within a unit distance, each class  $V_i$  of  $\mathcal{P}$  induces a clique in  $G$ . Moreover, the neighbors of a vertex belong to at most 14 classes of  $\mathcal{P}$ . This is because any unit disk intersects at most

14 squares; see Figure 4. In the example shown in Figure 4, the unit disk intersects the gray squares. The disk touches the square in the lower left, but we do not say that they intersect because a vertex on a border belongs to the upper-right square. Hence the unit disk graph satisfies property  $\pi$  with  $\theta = 14$ .  $\square$

In the remainder of this section, we present an algorithm for the connected dominating set problem, assuming that the input graph  $G$  possesses property  $\pi$  for some constant  $\theta$ . We also assume that a vertex  $r$  is spanned by the tree in an optimal solution to the problem. Of course, we do not know which vertex in  $G$  is spanned, but we can guess it by applying the algorithm with setting each vertex in  $V$  to  $r$ .

For each  $v \in V$ , let  $\mathcal{P}_v$  be the set of paths between  $r$  and  $v$ . Under the assumption that  $r$  is spanned by an optimal solution, the problem is relaxed to the following LP:

$$\begin{aligned}
& \text{minimize} && \sum_{v \in V} w(v)x(v) + \sum_{e \in E} l(e)y(e) \\
& \text{subject to} && \sum_{v \in N_G[t]} x(v) \geq 1 && \forall t \in T, \\
& && \sum_{P \in \mathcal{P}_v} f(P) = x(v) && \forall v \in V, \\
& && \sum_{P \in \mathcal{P}_v: e \in P} f(P) \leq y(e) && \forall e \in E, \forall v \in V, \\
& && x(v) \geq 0 && \forall v \in V, \\
& && y(e) \geq 0 && \forall e \in E, \\
& && f(P) \geq 0 && \forall P \in \bigcup_{v \in V} \mathcal{P}_v.
\end{aligned} \tag{1}$$

Indeed, if  $x \in \{0, 1\}^V$  and  $y \in \{0, 1\}^E$ , then the feasible solution  $(x, y, f)$  for (1) corresponds to a feasible solution to the connected dominating set problem. Here,  $x(v)$  indicates if vertex  $v$  is included in a dominating set  $S$  (if  $x(v) = 1$ ,  $v$  is included in  $S$ ), and  $y(e)$  indicates if edge  $e$  is included in a tree  $F$  that spans the dominating set  $S$  (if  $y(e) = 1$ ,  $e$  is included in  $F$ ). Also,  $f(P)$  represents the flow value along path  $P$ . The second constraint demands that the flow value between  $r$  and  $v$  is at least  $x(v)$ , and the third constraint means that the flow between  $r$  and  $v$  obeys the edge capacities  $y$ . Hence, if  $x(v) = 1$ , one unit of flow runs between  $r$  and  $v$ . This means that the minimum cut separating  $v$  from  $r$  with respect to the edge capacities  $y$  has a capacity of at least 1. Hence the edge set  $\{e \in E: y(e) = 1\}$  connects  $r$  and each vertex  $v$  with  $x(v) = 1$ .

Although there are an exponential number of variables in the LP (1), it can be converted into an equivalent formulation of polynomial size. Hence an optimal solution  $(x^*, y^*, f^*)$  for (1) can be computed in polynomial time. Our algorithm computes this, and then from this optimal fractional solution, it constructs a dominating set  $S$  and a tree  $F$ , as follows.

We define  $I$  as  $\{i = 1, \dots, k: \sum_{v \in V_i} x^*(v) \geq 1/\theta\}$ . We restrict a dominating set to be included in  $V_I := \bigcup_{i \in I} V_i$ . Namely, the dominating set  $S$  computed by our algorithm is feasible to the following integer program:

$$\begin{aligned}
& \text{minimize} && \sum_{v \in V} w(v)x(v) \\
& \text{subject to} && \sum_{v \in N[t] \cap V_I} x(v) \geq 1 && \forall t \in T, \\
& && x(v) \in \{0, 1\} && \forall v \in V.
\end{aligned} \tag{2}$$

By replacing constraint  $x(v) \in \{0, 1\}$  with  $x(v) \geq 0$ , we obtain an LP relaxation of (2). By the following lemma, the optimal objective value of this relaxation can be bounded by  $\theta$  times the weight of  $x^*$ .

**Lemma 4.** *The LP relaxation of (2) has the optimal objective value that is at most  $\theta \sum_{v \in V} w(v)x^*(v)$ .*

*Proof.* For each terminal  $t \in T$ , there is a class  $V_i$  of  $\mathcal{P}$  with  $\sum_{v \in N[t] \cap V_i} x^*(v) \geq 1/\theta$  (and hence  $i \in I$ ), because  $\sum_{v \in N[t]} x^*(v) \geq 1$  holds and the vertices in  $N[t]$  belong to at most  $\theta$  classes of partition  $\mathcal{P}$ . This implies that  $\theta x^*$  is feasible to the LP relaxation of (2). Therefore, the optimal objective value of the relaxation is at most  $\theta \sum_{v \in V} w(v)x^*(v)$ .  $\square$

Problem (2) is a special case of the geometric set cover problem, in which the ground set is a set of points on a Euclidean plane, and each set is represented by a unit disk. There is a constant-factor LP rounding algorithm for this problem that uses the LP relaxation of (2) [4]. Let  $\gamma$  be the approximation factor of this algorithm. Then, this algorithm computes  $S \subseteq V_I$  such that  $T$  is dominated by  $S$  and  $w(S) \leq \gamma \theta \sum_{v \in V} w(v)x^*(v)$ . In our algorithm for the connected dominating set problem, the dominating set is defined as the vertex set  $S$  computed by the algorithm for the geometric set cover problem.

Our algorithm then computes a tree that spans  $r$  and  $S$ . Let us explain how to compute the tree. For each  $i \in I$ , we choose an arbitrary vertex in  $V_i$  and call it  $v_i$ . We use an algorithm for the Steiner tree problem to construct a minimum-length tree that spans  $r$  and all vertices  $v_i, i \in I$ . An LP relaxation of this Steiner tree problem can be written as follows:

$$\begin{aligned}
& \text{minimize} && \sum_{e \in E} l(e)y(e) \\
& \text{subject to} && \sum_{P \in \mathcal{P}_{v_i}} f(P) = 1 && \forall i \in I, \\
& && \sum_{P \in \mathcal{P}_{v_i}: e \in P} f(P) \leq y(e) && \forall e \in E, \forall i \in I, \\
& && y(e) \geq 0 && \forall e \in E, \\
& && f(P) \geq 0 && \forall P \in \bigcup_{i \in I} \mathcal{P}_{v_i}.
\end{aligned} \tag{3}$$

We note that  $(y^*, f^*)$  is not necessarily feasible to (3). Nevertheless, we can bound the optimal objective value of (3).

**Lemma 5.** *The optimal objective value of (3) is at most  $\theta(\sum_{v \in V} w(v)x^*(v) + \sum_{e \in E} l(e)y^*(e))$ .*

*Proof.* We define a feasible solution for (3) from  $(y^*, f^*)$ . First, initialize  $(y, f)$  to  $(y^*, f^*)$ . Let  $i \in I$ . Then,  $\sum_{v \in V_i} \sum_{P \in \mathcal{P}_v} f^*(P) \geq \sum_{v \in V_i} x^*(v) \geq 1/\theta$ . Recall that there exists an edge  $vv_i$  for each vertex  $v \in V_i \setminus v_i$ . If  $\sum_{P \in \mathcal{P}_v} f^*(P) = \epsilon$  for  $v \in V_i \setminus v_i$ , we increase  $y(vv_i)$  by  $\epsilon$ , increase  $f(P \cup \{vv_i\})$  by  $f(P)$ , and set  $f(P) = 0$ , for every  $P \in \mathcal{P}_v$ . Notice that  $l(vv_i)$  does not exceed  $w(v)$ . Hence the increase of  $y(vv_i)$  costs  $l(vv_i)\epsilon \leq w(v)\epsilon = w(v)x^*(v)$ . We do this for every  $i \in I$  and for every vertex  $v \in V_i \setminus v_i$ . At the termination of this procedure,  $\sum_{e \in E} l(e)y(e) \leq \sum_{v \in V} w(v)x^*(v) + \sum_{e \in E} l(e)y^*(e)$ , and  $\sum_{P \in \mathcal{P}_{v_i}} f(P) = \sum_{v \in V_i} \sum_{P \in \mathcal{P}_v} f^*(P) \geq 1/\theta$ . We define  $(y', f')$  as  $(\theta y, \theta f)$ . Then,  $(y', f')$  is feasible for (3), and its objective value does not exceed  $\theta(\sum_{v \in V} w(v)x^*(v) + \sum_{e \in E} l(e)y^*(e))$ , completing the proof.  $\square$

Goemans and Bertsimas [10] showed that a Steiner tree of length at most twice the optimal objective value of (3) can be computed from a minimum spanning tree in the metric completion on the terminal set. Namely, there is an algorithm that computes a tree  $F$  spanning  $r$  and all vertices  $v_i, i \in I$ , such that  $l(F) \leq 2\theta(\sum_{v \in V} w(v)x^*(v) + \sum_{e \in E} l(e)y^*(e))$ .  $F$  may not span a vertex  $v \in S$ . For such a vertex  $v$ , we add an edge joining  $v$  with  $v_i$ , where  $i$  is an index such that  $v \in V_i$ . Notice that  $G$  contains an edge  $vv_i$ , because  $V_i$  induces a clique. Let  $F'$  denote the set of these added edges. Notice that  $l(F') \leq w(S)$ . Our algorithm outputs  $(F \cup F', S)$  as a solution for the connected dominating set problem. Recall that we are assuming here that  $r$  is spanned by an optimal solution. When we implement the algorithm, we apply it to the vertices in  $V$  as  $r$ , and define the output as the best of the obtained solutions.

**Theorem 6.** *The solution  $(F \cup F', S)$  computed by the above algorithm is a  $2(\gamma + 1)\theta$ -approximate solution for the connected dominating set problem.*

*Proof.*  $S$  dominates  $T$ . Moreover,  $F$  connects each vertex  $v_i$ ,  $i \in I$ , to  $r$ , and  $F'$  connects each vertex  $v \in S \cap V_i$  to  $v_i$ . Hence  $(F \cup F', S)$  is feasible for the connected dominating set problem.

As noted above, we have  $l(F') \leq w(S) \leq \gamma\theta \sum_{v \in V} w(v)x^*(v)$ , and  $l(F) \leq 2\theta(\sum_{v \in V} w(v)x^*(v) + \sum_{e \in E} l(e)y^*(e))$ . Hence the objective value  $l(F) + l(F') + w(S)$  is at most  $2(\gamma + 1)\theta$  times the optimal objective value of (1). Since (1) relaxes the connected dominating set problem, this proves the theorem.  $\square$

Recall that  $\gamma$  and  $\theta$  are constants if the graph is a unit disk graph. Hence, Theorem 6 has the following corollary.

**Corollary 2.** *The VC-weighted Steiner tree problem admits a constant-factor approximation algorithm in unit disk graphs.*

## 5 Steiner tree activation problem in graphs excluding a fixed minor

In this section, we present a constant-factor approximation algorithm for the Steiner tree activation problem in graphs excluding a fixed minor. In particular, our algorithm is a 11-approximation for planar graphs.

Our algorithm is based on the reduction mentioned in Theorem 1. We reduce the problem to the vertex-weighted Steiner tree problem by using that reduction, and we solve the obtained instance by using the constant-factor approximation algorithm proposed by Demaine, Hajiaghayi, and Klein [6] for the vertex-weighted Steiner tree problem in graphs excluding a fixed minor. We prove that this achieves a constant-factor approximation for the Steiner tree activation problem when the input graph is  $H$ -minor-free for some graph  $H$  such that  $|V(H)|$  is a constant.

This seems to be an easy corollary to Demaine et al., but it is not so because the reduction does not preserve the  $H$ -minor-freeness of the input graph. Let  $G$  be the graph obtained by removing one edge from  $K_5$ . It is easy to check that  $G$  is planar. We consider the VC-weighted spanning tree problem over  $G$ . The reduction transforms  $G$  into another graph  $G'$  on the vertex set  $V(G) \cup \{v_\circ, v_\bullet : v \in V(G)\}$ . Refer to the proof of Theorem 1 for the definition of the edge set of  $G'$ . Notice that the subgraph of  $G'$  induced by  $V_\bullet := \{v_\bullet : v \in V(G)\}$  is isomorphic to  $G$ . Let  $u$  be an arbitrary vertex in  $V(G)$  that is not an end vertex of the removed edge. The subgraph of  $G'$  induced by  $V_\bullet \cup u_\circ$  contains a subgraph isomorphic to a subdivision of  $K_5$ , and hence  $G'$  is not planar.

As indicated by this example, the reduction does not preserve the  $H$ -minor-freeness. In spite of this, we can prove that the approximation guarantee given by Demaine et al. extends to the graphs constructed from a  $H$ -minor-free graph by the reduction.

We recall that the reduction constructs a graph  $G'$  on the vertex set  $T \cup \{v_i : v \in V, i \in W\}$  from the input graph  $G = (V, E)$  and the monotone activation functions  $f_{uv} : W \times W \rightarrow \{\top, \perp\}$ ,  $uv \in E$ . We denote the vertex set  $\{v_i : i \in W\}$  defined from an original vertex  $v \in V$  by  $U_v$ . Let  $U$  denote  $\bigcup_{v \in V} U_v$ .

First, let us illustrate how the algorithm of Demaine et al. behaves for  $G'$ . The algorithm maintains a vertex set  $X \subseteq T \cup U$ , where  $X$  is initialized to  $T$  at the beginning. Let  $\mathcal{A}(X) \subseteq 2^X$  denote the family of connected components that include some terminals in the subgraph of  $G'[X]$ . We call each member of  $\mathcal{A}(X)$  an *active set*. The algorithm consists of two phases, called the increase phase and the reverse-deletion phase. In the increase phase, the algorithm iteratively adds vertices to  $X$  until  $|\mathcal{A}(X)|$  is equal to one. This implies that, when the increase phase terminates, the subgraph induced by  $X$  connects all of the terminals. In the reverse-deletion phase,  $X$  is transformed into an inclusion-wise minimal vertex set that induces a Steiner tree. This is done by repeatedly removing vertices from  $X$  in the reverse of the order in which they were added.

Let  $\bar{X}$  be the vertex set  $X$  when the algorithm terminates, and let  $X$  be the vertex set at some point during the increase phase. We denote  $\bar{X} \setminus X$  by  $\bar{X}'$ . Note that  $\bar{X}'$  is a minimal augmentation of  $X$  such that  $X \cup \bar{X}'$  induces a Steiner tree. Each  $Y \in \mathcal{A}(X)$  is disjoint from  $\bar{X}'$ , because  $Y \subseteq X$ . Demaine et al. showed the following analysis of their algorithm.

**Theorem 7** ([6]). *Let  $X$  be a vertex set maintained at some moment in the increase phase, and let  $\bar{X}'$  be a minimal augmentation of  $X$  so that  $X \cup \bar{X}'$  induces a Steiner tree. If there is a number  $\gamma$  such that  $\sum_{Y \in \mathcal{A}(X)} |\bar{X}' \cap N(Y)| \leq \gamma |\mathcal{A}(X)|$  holds for any  $X$  and  $\bar{X}'$ , the algorithm of Demaine et al. achieves an approximation factor  $\gamma$ .*

In  $G'[\bar{X}' \cup (\bigcup_{Y \in \mathcal{A}(X)} Y)]$ , contract each  $Y \in \mathcal{A}(X)$  into a single vertex, discard all edges induced by  $\bar{X}'$  and all isolated vertices in  $\bar{X}'$ , and replace multiple edges by single edges. This gives us a simple bipartite graph with the bipartition  $\{A, B\}$  of the vertex set, where each vertex in  $A$  corresponds to an active set, and  $B$  is a subset of  $\bar{X}'$ . Let  $D$  denote this graph. This construction of  $D$  is illustrated in Figure 5. We note that  $\sum_{Y \in \mathcal{A}(X)} |\bar{X}' \cap N(Y)|$  is equal to the number of edges in  $D$ . Hence, by Theorem 7, if the number of edges is at most a constant factor of  $|A|$ , the algorithm achieves a constant-factor approximation.

Demaine et al. proved that  $|B| \leq 2|A|$ , and  $D$  is  $H$ -minor-free if  $G$  is  $H$ -minor-free. By [15, 20], these two facts imply that the number of edges in  $D$  is  $O(|A||V(H)|\sqrt{\log |V(H)|})$ . When  $G$  is planar, together with Euler's formula and the fact that  $D$  is bipartite, they imply that the number of edges in  $D$  is at most  $6|A|$ .

The proof of Demaine et al. for  $|B| \leq 2|A|$  can be carried to our case. However,  $D$  is not necessarily  $H$ -minor-free even if  $G$  is  $H$ -minor-free. Nevertheless, we can bound the number of edges in  $D$ , as follows.

**Lemma 6.** *Suppose that the given activation function is monotone. If  $G$  is  $H$ -minor-free, the number of edges in  $D$  is  $O(|A||V(H)|\sqrt{\log |V(H)|})$ . If  $G$  is planar, the number of edges in  $D$  is at most  $11|A|$ .*

The following theorem is immediate from Theorem 7 and Lemma 6.

**Theorem 8.** *If an input graph is  $H$ -minor-free for some graph  $H$ , then the Steiner tree activation problem with a monotone activation function admits an  $O(|V(H)|\sqrt{\log |V(H)|})$ -approximation algorithm. In particular, if the input graph is planar, then the problem admits a 11-approximation algorithm.*

In the rest of this section, we prove Lemma 6. We first provide several preparatory lemmas.

**Lemma 7.** *If  $G'$  includes an edge  $u_i v_j$  for some  $u, v \in V$  and  $i, j \in W$ , then  $G'$  also includes an edge  $u_{i'} v_{j'}$  for any  $i', j' \in W$  with  $i' \geq i$  and  $j' \geq j$ .*

*Proof.* The lemma is immediate from the construction of  $G'$  and the assumption that each edge in  $G$  is associated with a monotone activation function.  $\square$

**Lemma 8.**  *$\bar{X}$  does not contain any two distinct copies of an original vertex.*

*Proof.* For the sake of a contradiction, suppose that  $v_i, v_j \in \bar{X}$  for some  $v \in V$  and  $i, j \in W$  with  $i < j$ . If an edge  $u_k v_i$  exists in  $G'$ , then another edge  $u_k v_j$  also exists by Lemma 7. This means that  $\bar{X} \setminus v_i$  induces a Steiner tree in  $G'$ , which contradicts the minimality of  $\bar{X}$ .  $\square$

**Lemma 9.** *Let  $Y, Y' \in \mathcal{A}(X)$  with  $Y \neq Y'$ . If  $Y \cap U_v \neq \emptyset$  for some  $v \in V$ , then  $Y' \cap U_v = \emptyset$ .*

*Proof.* Suppose that  $Y \cap U_v \neq \emptyset \neq Y' \cap U_v$ . Let  $v_i \in Y$  and  $v_j \in Y'$  with  $i < j$ . A vertex adjacent to  $v_i$  is also adjacent to  $v_j$  in  $G'$  by Lemma 7. By the definition,  $Y$  induces a connected component of  $G'[X]$  that includes a terminal  $t$ . Hence  $v_i$  has at least one neighbor in  $Y$ . This implies that  $v_i$  and  $v_j$  are connected in  $G'[X]$ . This contradicts the fact that  $Y$  and  $Y'$  are different connected components of  $G'[X]$ .  $\square$

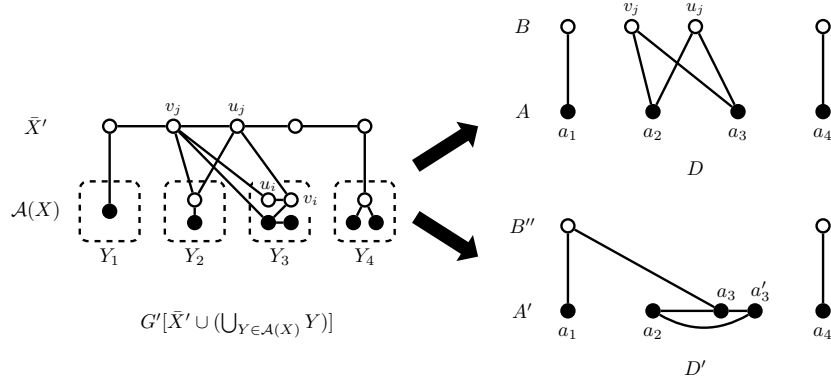


Figure 5: An example of  $G'[\bar{X}' \cup (\bigcup_{Y \in \mathcal{A}(X)} Y)]$ ,  $D$ , and  $D'$ ; in construction of  $D'$ ,  $\bar{Y}_3$  is divided into two subsets, one of which contains  $u_j$  and the other contains  $v_j$ ; the former is shrunk into  $a_3$  and the later is shrunk into  $a'_3$

Consider  $Y \in \mathcal{A}(X)$  and  $v \in V$  such that  $Y \cap U_v \neq \emptyset$ . Let  $v_i$  be the vertex that has the largest subscript in  $Y \cap U_v$  (i.e.,  $i = \max\{i' \in W : v_{i'} \in Y \cap U_v\}$ ). Then, from  $Y$ , we remove all vertices in  $Y \cap U_v$  but  $v_i$ . Moreover, if a copy  $v_j$  of  $v$  is included in  $B$ , we replace  $v_i$  by  $v_j$ . Notice that  $j > i$  holds in this case by Lemma 7, and  $B$  does not include more than one copy of  $v$  because of Lemma 8. Let  $\bar{Y}$  denote the vertex set obtained from  $Y$  by doing these operations for each  $v \in V$  with  $Y \cap U_v \neq \emptyset$ .  $\bar{Y}$  induces a connected subgraph of  $G'$  because of Lemma 7.

We let  $V_B$  denote  $\{v \in V : B \cap U_v \neq \emptyset\}$ , and let  $V_{B,Y}$  denote  $\{v \in V_B : \bar{Y} \cap U_v \neq \emptyset\}$  for each  $Y \in \mathcal{A}(X)$ . Moreover, let  $B'$  denote  $B \cap \{U_v : v \in \bigcup_{Y \in \mathcal{A}(X)} V_{B,Y}\}$ , and  $B''$  denote  $B \setminus B'$ . In other words, each vertex  $v_j \in B$  belongs to  $B'$  if and only if some copy  $v_i$  of the same original vertex  $v \in V$  is contained by an active set in  $\mathcal{A}(X)$ .

If  $k := |V_{B,Y}| \geq 2$ , we divide  $\bar{Y}$  into  $k$  subsets such that the copies of the vertices in  $V_{B,Y}$  belong to different subsets, and each subset induces a connected subgraph of  $G'$ . Let  $\mathcal{A}'(X)$  denote the family of vertex sets obtained by doing these operations to all active sets in  $\mathcal{A}(X)$ . Notice that  $|\mathcal{A}'(X)| = |\mathcal{A}(X)| + \sum_{Y \in \mathcal{A}(X)} \max\{0, |V_{B,Y}| - 1\}$ . Lemma 9 indicates that, if a vertex  $v \in V_B$  belongs to  $V_{B,Y}$  for some  $Y \in \mathcal{A}(X)$ , then it does not belong to  $V_{B,Y'}$  for any  $Y' \in \mathcal{A}(X) \setminus \{Y\}$ . Thus,  $\sum_{Y \in \mathcal{A}} |V_{B,Y}| \leq |B'|$ , and hence  $|\mathcal{A}'(X)| \leq |\mathcal{A}(X)| + |B'|$ .

We shrink each  $Z \in \mathcal{A}'(X)$  into a single vertex in the induced subgraph  $G'[B'' \cup (\bigcup_{Z \in \mathcal{A}'(X)} Z)]$  of  $G'$ , and convert the obtained graph into a simple graph by removing all self-loops and by replacing multiple edges with single edges. Let  $A'$  denote the set of vertices obtained by shrinking vertex sets in  $\mathcal{A}'(X)$ , and let  $D'$  denote the obtained graph (with the vertex set  $A' \cup B''$ ). See Figure 5 for an illustration of this construction. We observe that  $D'$  is  $H$ -minor-free in the following lemma.

**Lemma 10.** *If  $G$  is  $H$ -minor-free, then  $D'$  is  $H$ -minor-free.*

*Proof.* By Lemma 8 and the construction of  $\mathcal{A}'(X)$ , each vertex in  $V$  has at most one copy in  $B'' \cup (\bigcup_{Z \in \mathcal{A}'(X)} Z)$ . If  $G'[B'' \cup (\bigcup_{Z \in \mathcal{A}'(X)} Z)]$  includes an edge  $u_i v_j$  for  $u_i \in U_u$  and  $v_j \in U_v$ , then  $G$  also includes an edge  $uv$ . Thus  $G'[B'' \cup (\bigcup_{Z \in \mathcal{A}'(X)} Z)]$  is isomorphic to a subgraph of  $G$ . Since each  $Z \in \mathcal{A}'(X)$  induces a connected subgraph of  $G'$ , the graph  $D'$  (constructed from  $G'[B'' \cup (\bigcup_{Z \in \mathcal{A}'(X)} Z)]$  by shrinking each  $Z \in \mathcal{A}'(X)$ ) is a minor of  $G$ . Hence if  $G$  is  $H$ -minor-free,  $D'$  is also  $H$ -minor-free.  $\square$

The following lemma gives a relationship between  $D$  and  $D'$ .

**Lemma 11.** *If  $l$  is the number of edges in  $D'$ , then  $D$  contains at most  $l + |B'|$  edges.*

*Proof.* Let  $av_i$  be an edge in  $D$  that joins vertices  $a \in A$  and  $v_j \in B$ . Suppose that  $a$  is a vertex obtained by shrinking  $Y \in \mathcal{A}(X)$ , and  $v_j$  is a copy of  $v \in V$ . Remember that  $v_j$  belongs to either  $B'$  or  $B''$ . If  $v_j \in B'$ , it is contained by a vertex set in  $\mathcal{A}'(X)$ , denoted by  $Z_v$ . We consider the following three cases:

1.  $v_j \in B'$  and  $Z_v \subseteq \bar{Y}$
2.  $v_j \in B'$  and  $Z_v \not\subseteq \bar{Y}$
3.  $v_j \in B''$

In the second case, an edge in  $D'$  joins vertices obtained by shrinking  $Z_v$  and a subset of  $\bar{Y}$ . In the third case,  $v_j$  exists in  $D'$ , and  $D'$  includes an edge that joins  $v_j$  and the vertex obtained by shrinking a subset of  $\bar{Y}$ . Thus  $D'$  includes an edge corresponding to  $av_j$  in these two cases. We can also observe that no edge in  $D'$  corresponds to more than two such edges  $av_j$ . This is because  $Z_u \neq Z_v$  for any distinct vertices  $u_i$  and  $v_j$  in  $B'$  by the construction of  $\mathcal{A}'(X)$ .

In the first case,  $D'$  may not contain an edge corresponding to  $av_j$ . However, the number of such edges is at most  $|B'|$  in total because  $\bar{Y}$  are uniquely determined from  $v_j$  in this case. Therefore, the number of edges in  $D$  is at most  $l + |B'|$ .  $\square$

We now prove Lemma 6.

*Lemma 6.* The number of vertices in  $D'$  is at most  $|\mathcal{A}'(X)| + |B''| \leq |\mathcal{A}(X)| + |B'| + |B''| = |A| + |B|$ . As we mentioned, we can prove  $|B| \leq 2|A|$  similar to Demaine et al. [6]. Hence  $D'$  contains at most  $3|A|$  vertices. By Lemma 10,  $D'$  is  $H$ -minor-free. It is known [15, 20] that the number of edges in an  $H$ -minor-free graph with  $n$  vertices is  $O(n|V(H)|\sqrt{\log |V(H)|})$ . Therefore, the number of edges in  $D'$  is  $O(|A||V(H)|\sqrt{\log |V(H)|})$ . By Lemma 11, this implies that the number of edges in  $D$  is  $|B'| + O(|A||V(H)|\sqrt{\log |V(H)|}) = O(|A||V(H)|\sqrt{\log |V(H)|})$ . This fact and Theorem 7 prove the former part of the lemma.

If  $G$  is planar, by Euler's formula, the number of edges in  $D'$  is at most  $3(|A| + |B|)$ . Hence, by Lemma 11, the number of edges in  $D$  is at most  $3(|A| + |B|) + |B'| \leq 3|A| + 4|B| \leq 11|A|$ . The latter part of the lemma follows from this fact and Theorem 7.  $\square$

## 6 Conclusion

In this paper, we formulate the VC-weighted Steiner tree problem, a new variant of the vertex-weighted Steiner tree and the Steiner tree activation problems. We proved that it is NP-hard for unit disk graphs and planar graphs. We also presented constant-factor approximation algorithms for the VC-weighted Steiner tree problem with unit disk graphs and for the Steiner tree activation problem with graphs excluding a fixed minor.

An interesting future work is to investigate VC-weighted spanning tree or VC-weighted Steiner tree problem with unit weights for unit disk graphs and planar graphs. We do not know whether these problems are NP-hard or admits exact polynomial-time algorithms. Finding a constant-factor approximation algorithm for the Steiner tree activation problem with unit disk graphs also remains an open problem.

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